On the Equations and Properties—(1) of the System of Circles touching three Circles in a Plane; (2) of the System of Spheres touching four Spheres in Space; (3) of the System of Circles touching three Circles on a Sphere; (4) on the System of Conics inscribed to a Conic, and touching three inscribed Conics in a Plane.

In the following Paper I shall give—1°. The method of investigating the equations of the circles in pairs which touch three given circles in a plane; 2°. the equations of the spheres in pairs which touch four given spheres; 3°. the equations of the circles in pairs which touch three others on the surface of a sphere; 4°. the equations in pairs of the conics having double contact with a given conic which touch three other conics having also double contact with the same given conic.

In the course of the investigation I shall, besides giving the methods by which I discovered the equations, indicate other methods which subsequently occurred to me, and shall show that some of the results are but generalizations of equations with which geometers have been long familiar.

I.

Equations of the Circles which touch three others.

Art. 1.—If A, B, C, D, be four points on a line disposed in any manner, then always, none of the four being at infinity,

$$BC \cdot AD + CA \cdot BD + AB \cdot CD = 0, \tag{1}$$

regard being had to the signs as well as to the magnitude of the six segments involved—Townsend's "Modern Geometry," vol. i., p. 102. Now, let A, B, C, D, be the points of contact of four circles which touch the line, and whose diameters are δ, δ', δ'', δ''', then, dividing equation (1) by $\sqrt{\delta, \delta', \delta'', \delta'''}$, we get

$$\frac{BC}{\sqrt{\delta' \delta''}} \cdot \frac{AD}{\sqrt{\delta \delta'''}} + \frac{CA}{\sqrt{\delta'' \delta}} \cdot \frac{BD}{\sqrt{\delta' \delta'''}} + \frac{AB}{\sqrt{\delta \delta'}} \cdot \frac{CD}{\sqrt{\delta'' \delta'''}} = 0.$$

Now, since, if two circles be inverted from any arbitrary point, the ratio of the square of their common tangent to the rectangle contained by their diameters is constant, that is, remains unaltered by the inversion (See Townsend's "Modern Geometry," vol. ii., p. 375), each of the ratios

$$\frac{BC}{\sqrt{\delta' \delta''}}, \quad \frac{AD}{\sqrt{\delta \delta'''}}, \text{ \&c.,}$$

is unaltered by inversion. Hence we have immediately the following theorem, which is obviously an extension of Ptolemy's theorem concerning four points on a circle:—

"If S, S', S'', S''', be four circles which touch a fifth circle Σ, the common tangent of $S\,S'$ by the common tangent of $S''\,S'''$ plus the common tangent of S', S'' by the common tangent of S, S''', plus the common tangent of S'', S, by the common tangent of $S'\,S''' = 0$; the common tangent of any pair of circles being direct or transverse, according as their contacts with Σ are similar or dissimilar."

The foregoing theorem being the foundation of nearly the whole of the following Paper, I shall in a subsequent article give another proof of it, not derived, like the preceding, from inversion, and which, slightly modified, will give the corresponding theorem respecting four circles which touch a fifth circle on the surface of a sphere.

2. If we denote the direct common tangents to S', S''; S'', S; S, S'; by

$$l^{\frac{1}{2}},\ m^{\frac{1}{2}},\ n^{\frac{1}{2}},\ \text{respectively,}$$

and the transverse common tangents by

$$l'^{\frac{1}{2}},\ m'^{\frac{1}{2}},\ n'^{\frac{1}{2}};$$

and supposing the fourth circle S''' to become a point, the common tangents to S''', S; S''', S'; S''', S''; become the square roots of the results of substituting the co-ordinates of the point S''' in the equations of the circles S, S', S'', respectively, and hence they are

$$\sqrt{S},\quad \sqrt{S'},\quad \sqrt{S''}.$$

Hence, the co-ordinates of any point in the circle touching $S\,S'\,S''$ must satisfy the equation

$$\sqrt{lS} + \sqrt{mS'} + \sqrt{nS''} = 0; \tag{2}$$

and since this equation, cleared of radicals, is of the fourth degree, it is the equation of a pair of circles, Σ, Σ', touching S, S', S'', as in fig. (1).

In like manner, the equations of the three other pairs of circles touching S, S', S'', are

$$\sqrt{lS} + \sqrt{m'S'} + \sqrt{n'S''} = 0 \tag{3}$$

$$\sqrt{l'S} + \sqrt{mS'} + \sqrt{n'S''} = 0 \tag{4}$$

$$\sqrt{l'S} + \sqrt{m'S} + \sqrt{nS''} = 0 \tag{5}$$

3. The equations of the inscribed and exscribed circles of a plane triangle are particular cases of the equations (2), (3), (4), (5), and may be inferred from them as follows:—Let the radii of the circles S, S', S'', be a, b, c, and let the angles in which they intersect each other be—

for	S', S'';	A;
,,	S'', S;	B;
,,	S, S;	C;

then it is easy to see that

$$2 \cos \tfrac{1}{2} A = \sqrt{\frac{l}{bc}}$$

$$2 \cos \tfrac{1}{2} B = \sqrt{\frac{m}{ca}}$$

$$2 \cos \tfrac{1}{2} C = \sqrt{\frac{n}{ab}}$$

Substituting the values of $\sqrt{l}$, $\sqrt{m}$, $\sqrt{n}$, from these equations in equation (2), it becomes, by dividing by $2\sqrt{2abc}$,

$$\cos \tfrac{1}{2} A \sqrt{\frac{S}{2a}} + \cos \tfrac{1}{2} B \sqrt{\frac{S'}{2b}} + \cos \tfrac{1}{2} C \sqrt{\frac{S''}{2c}} = 0. \qquad (6)$$

Now, if S be a circle (fig. 2), and O a point whose co-ordinates are substituted in the equation of S, we have, if our equations be Cartesian,

$$\frac{S}{2a} = \frac{OP \cdot OQ}{PQ}.$$

And when the circle S becomes infinitely large, the limit of $OQ : PQ$ is unity. Hence the limits of $\frac{S}{2a}$, $\frac{S'}{2b}$, $\frac{S''}{2c}$, when S S' S'' becomes right lines, are the perpendiculars let fall on these lines, and denoting them by α, β, γ, the equation (6) becomes

$$\cos \tfrac{1}{2} A \sqrt{\alpha} + \cos \tfrac{1}{2} B \sqrt{\beta} + \cos \tfrac{1}{2} C \sqrt{\gamma} = 0. \qquad (7)$$

This is the equation of the circle inscribed in the triangle formed by the lines α, β, γ; and the exscribed circles may in like manner be derived from the equations (3), (4), (5).—Q. E. D.

4. The equation of the circle circumscribed about a triangle is also a particular case of equation (2). Thus, let S, S', S'', become points, denoting them by A, B, C, and the point S''' by D, then equation (2) becomes Ptolemy's theorem,

$$BC \cdot AD + CA \cdot BD + AB \cdot CD = 0.$$

Hence,

$$\frac{BC}{BD \cdot CD} + \frac{CA}{CD \cdot AD} + \frac{AB}{AD \cdot BD} = 0. \qquad (8)$$

Now, BC, CA, AB, are proportional to sin A, sin B, sin C; and if the equations of the lines BC, CA, AB, be α, β, γ, we have

$$BD\cdot CD = \alpha \cdot \text{diameter of circle};$$

$$CD\cdot AO = \beta \cdot \text{diameter of circle};$$

$$AD\cdot BD = \gamma \cdot \text{diameter of circle}.$$

Hence equation (8) becomes

$$\frac{\sin A}{\alpha} + \frac{\sin B}{\beta} + \frac{\sin C}{\gamma} = 0. \tag{9}$$

This is the equation of the circumscribed circle.

5. The equation of the inscribed circle of a plane triangle has been derived in Art. 3 from the equation (2) of a pair of circles touching three circles. Conversely, the equation of a pair of circles touching three others may be derived from the equation of the inscribed circle of a plane triangle.

For let Σ (fig. 3) be the circle inscribed in the triangle ABC, the equations of whose sides are $\alpha = 0$; $\beta = 0$; $\gamma = 0$; and let the circles S, S', S'', touch Σ at its points of contact with the sides of the triangle ABC; then denoting the radius of Σ by R, and the radii of S, S', S'', by r, r', r'', respectively; if any point Q be taken in Σ, the result of substituting the co-ordinates of Q in $S = 2(R-r)$ multiplied by the result of substituting the co-ordinates Q in α.

This may be written

$$\alpha = \frac{S}{2(R-r)};$$

in like manner,

$$\beta = \frac{S'}{2(R-r')};$$

and

$$\gamma = \frac{S''}{2(R-r'')}.$$

Again, since l denotes the direct common tangent to S', S'', it is easy to see that

$$\cos \tfrac{1}{2} A = \sqrt{\frac{l}{(R-r')(R-r'')}}.$$

In like manner,

$$\cos \tfrac{1}{2} B = \sqrt{\frac{m}{(R - r'')(R - r)}};$$

and

$$\cos \tfrac{1}{2} C = \sqrt{\frac{n}{(R - r)(R - r')}};$$

making these substitutions, the equation

$$\cos \tfrac{1}{2} A \sqrt{\alpha} + \cos \tfrac{1}{2} B \sqrt{\beta} + \cos \tfrac{1}{2} C \sqrt{\gamma} = 0$$

becomes transformed into

$$\sqrt{lS} + \sqrt{mS'} + \sqrt{nS''} = 0.$$

The equations of the other pairs of circles may be similarly derived from the equations of the exscribed circles.—Q. E. D.

6. The equation

$$\sqrt{lS} + \sqrt{mS'} + \sqrt{nS''} = 0,$$

when cleared of radicals, becomes

$$l^2 S^2 + m^2 S'^2 + n^2 S''^2 - 2lmSS' - 2mnS'S'' - 2nlS''S = 0 \qquad (10)$$

Now, since this may be written in either of the equivalent forms

$$(lS - mS')^2 + nS''(nS'' - 2lS - 2mS') \qquad (11)$$

$$(mS' - nS'')^2 + lS(lS - 2mS' - 2nS'') \qquad (12)$$

$$(nS'' - lS)^2 + mS'(mS' - 2nS'' - 2lS) \qquad (13)$$

it follows that the pairs of circles

$$\sqrt{lS} + \sqrt{mS'} + \sqrt{nS''} = 0$$

touch S at the points $S = 0$, $mS' - nS'' = 0$;

,, S' ,, $S' = 0$, $nS'' - lS = 0$;

,, S'' ,, $S'' = 0$, $lS - mS' = 0$.

Hence, we have the following method of constructing the points of contact on the circles S, S', S'', with a pair of their tangential circles:—

Describe the circle $lS - mS' = 0$; this circle, coaxal with S, S', will intersect S'' in two points, which will be points of contact. Again, describe the circle $mS' - nS''$; it will intersect S in the points of contact. Lastly, describe the circle $nS'' - lS = 0$, and it will intersect S' in the points of contact.

7. Since the circle $lS - mS' = 0$ intersects $S'' = 0$ in the points of contact of S'' with the pair of circles $\sqrt{lS} + \sqrt{mS'} + \sqrt{nS''} = 0$,

$$lS - mS' - (l - m)\, S'' = 0$$

passes through the points of contact.

Now, $S - S'' = 0$ is the radical axis of the circles S, S'', and $S' - S'' = 0$ is the radical axis of the circles S', S''. Hence, denoting the radical axis of

$$\begin{array}{llcl} S' & S'' & \text{by} & A, \\ S'' & S & ,, & A', \\ S & S' & ,, & A'', \end{array}$$

this equation becomes

$$mA - lA' = 0;$$

hence,

$$\frac{A}{l} - \frac{A'}{m} = 0.$$

In like manner, the points of contact on S are constructed by drawing the line

$$\frac{A'}{m} - \frac{A''}{n} = 0;$$

and the points on S' by drawing the line

$$\frac{A''}{n} - \frac{A}{l} = 0.$$

Hence we derive the following theorem :—

The chords of contact of the three circles S, S', S'', with their four pairs of tangential circles, are given by the four systems of equations—

$$\frac{A}{l} = \frac{A'}{m} = \frac{A''}{n} \tag{14}$$

$$\frac{A}{l} = \frac{A'}{m'} = \frac{A''}{n'} \tag{15}$$

$$\frac{A}{l'} = \frac{A'}{m} = \frac{A''}{n'} \tag{16}$$

$$\frac{A}{l'} = \frac{A'}{m'} = \frac{A''}{n}. \tag{17}$$

8. Since the discriminant of the equation (10) does not vanish, it follows that it is not the product of two simple factors of the form

$$\lambda S + \mu S' + \nu S'' = 0;$$

$$\lambda' S + \mu' S' + \nu' S'' = 0.$$

Hence the equation of a circle (Σ) touching three circles, S, S', S'', cannot be expressed in the form $\lambda S + \mu S' + \nu S'' = 0$.

9. The result of Art. 8 may be proved independently, as follows, and we can thence infer, conversely, that the discriminant of equation (10) ought not to vanish:—

For, if possible, let the equation of a circle (Σ) touching three circles S, S', S'', be of the form $\lambda S + \mu S' + \nu S'' \equiv \Sigma$.
Hence,

$$\lambda S + \mu S' \equiv \Sigma - \nu S''.$$

Now, since Σ touches S'', the circle $\lambda S + \mu S' = 0$, which is coaxal with S and S', also touches S'' at its point of contact with Σ; but we have seen (Art. 6) that the circle coaxal with S and S', which passes through the point of contact of Σ with S'', cuts S'', instead of touching it. Hence the equation of a circle touching S, S', S'', cannot be of the form $\lambda S + \mu S' + \nu S'' = 0$.—Q. E. D.

This conclusion accords with the fact that three circles, S, S', S'', being given, the form $\lambda S + \mu S' + \nu S'' = 0$ is not sufficiently general to express the equation of any fourth circle. For the equation of any circle contains three independent constants, while $\lambda S + \mu S' + \nu S'' = 0$ contains but two, viz., the ratios $\lambda : \nu$ and $\mu : \nu$.

10. The equations (11), (12), (13), of Art. 6, being all of the form $R^2 = LM$, hence the pair of circles $\sqrt{lS} + \sqrt{mS'} + \sqrt{nS''}$ also touches the circles

$$lS - 2mS' - 2nS'' = 0;$$

$$mS' - 2nS'' - 2lS = 0;$$

$$nS'' - 2lS - 2mS' = 0.$$

Again, the circle $lS + mS' + nS''$ evidently passes through the intersections of the pairs of circles

$$S \text{ and } lS - 2mS' - 2nS'';$$

$$S' \text{ and } mS' - 2nS'' - 2lS;$$

$$S'' \text{ and } nS'' - 2lS - 2mS';$$

and is therefore coaxal with each pair. Hence the three lines joining the centres of these pairs of circles are concurrent. Hence we have the following theorem:—

The pair of circles $\sqrt{lS} + \sqrt{mS'} + \sqrt{nS''} = 0$ touching three circles, S, S', S'', also touches the three other circles

$$lS - 2mS' - 2nS'' = 0;$$
$$mS' - 2nS'' - 2lS = 0;$$
$$nS'' - 2lS - 2mS' = 0;$$

and the lines joining the centres of these circles to the centres of S, S', S'', respectively, concur to the centre of $lS + mS' + nS''$.

11. Since the equation (10) may be written in either of the following equivalent forms,

$$(lS + mS' - nS'')^2 = 4lmSS'; \tag{18}$$
$$(mS' + nS'' - lS)^2 = 4mn'SS''; \tag{19}$$
$$(nS'' + lS - mS')^2 = 4nlS''S; \tag{20}$$

we have the following theorem:—

The equations of the circles passing through the points of contact of the pair of circles $\sqrt{lS} + \sqrt{mS'} + \sqrt{nS''} = 0$ with

$$S, S' \text{ is } lS + mS' - nS'' = 0; \tag{21}$$
$$S', S'' \text{ ,, } mS' + nS'' - lS = 0; \tag{22}$$
$$S'' S \text{ ,, } nS'' + lS - Sm'' = 0. \tag{23}$$

12. We shall conclude this part of the subject of this Paper by applying our principles to prove Dr. Hart's celebrated extension of Feuerbach's theorem:—

"Taking any three of the eight circles which touch three others, a circle can be described to touch these three, and to touch a fourth circle of the eight touching circles."

Now, since the combinations in threes of eight things is $\frac{8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3} = 56$, this theorem makes it necessary that we should have fourteen circles, which we divide into two systems of circles—one system containing eight circles, and the other containing six.

Since the eight circles which touch three given circles are, four of them, the inverse of the other four, with respect to the circle which cuts the three given circles orthogonally, they may be denoted in pairs thus:—

$$\alpha\alpha'; \quad \beta\beta'; \quad \gamma\gamma'; \quad \delta\delta';$$

then it is evident that

$$\sqrt{lS} + \sqrt{mS'} + \sqrt{nS''} \equiv \alpha\alpha'$$
$$\sqrt{lS} + \sqrt{m'S'} + \sqrt{n'S''} \equiv \beta\beta'$$
$$\sqrt{l'S} + \sqrt{m'S'} + \sqrt{nS''} \equiv \gamma\gamma'$$
$$\sqrt{l'S} + \sqrt{mS'} + \sqrt{n'S''} \equiv \delta\delta'$$

Now, the equation of the pair of circles $\alpha\alpha'$, when cleared of radicals, is by equation (18)

$$4\, lmSS' = (lS + mS' - nS'')^2;$$

and this being of the form $LM = R^2$, the equation of any circle touching α and α' will be of the form

$$\mu^2 L - 2\mu R + M = 0$$

(Salmon's "Conic Sections," p. 234, Fourth Edition); or, restoring the values of L, M, R,

$$(\mu^2 - \mu) lS - (\mu - 1) mS' + \mu n S'' = 0. \qquad (a)$$

Similarly, the equation of any circle touching the pair of circles $\beta\beta'$ will be of the form

$$(\mu'^2 - \mu') lS - (\mu' - 1) m'S' + \mu' n' S'' = 0. \qquad (b)$$

In order that equations (a) and (b) may represent the same circle, we must have

$$\frac{\mu}{\mu'} = \frac{m}{m'}$$
$$\frac{\mu - 1}{\mu' - 1} = \frac{n}{n'}.$$

for the system of six circles.

Hence μ and μ' are determined, which proves the proposition; and we have the following system of six circles:—

$\alpha\alpha'\beta\beta'$	are all touched by a fourth circle.	(a)
$\alpha\alpha'\gamma\gamma'$	,,	(b)
$\alpha\alpha'\delta\delta'$	,,	(c)
$\beta\beta'\gamma\gamma'$	,,	(d)
$\beta\beta'\delta\delta'$	,,	(e)
$\gamma\gamma'\delta\delta'$	,,	(f)

And we see that from the relation between the system of eight circles,

$$\alpha,\ \beta,\ \gamma,\ \delta;\quad \alpha',\ \beta',\ \gamma',\ \delta';$$

and six circles,

$$(a),\ (b),\ (c),\ (d),\ (e),\ (f),$$

every circle of the former system is touched by three of the latter, and every circle of the latter by four of the former.—Q. E. D.

13. A very simple geometrical demonstration can be given of this part of Dr. Hart's theorem;—in fact, it is inferred at once from the following principle, which occurred to me some time since :—

If two circles, P', Q', be the inverse of two other circles, P, Q, with respect to the same circle, X, the four circles, P, Q, P', Q', have four common tangential circles.

This is evident.

14. Proof for the system of eight circles :—Let S, S', S'' (fig. 4), be the three given circles, any or all of which may be right lines, and α, β, γ, δ', four circles described touching them similar to the exscribed and inscribed circles of a plane triangle, I say, α, β, γ, δ', are all touched by a fourth circle, besides the three circles S, S,' S''.

Demonstration.—Let the direct common tangent to a pair of circles, α and β, for instance, be denoted by the notation $\alpha\beta$, and the transverse commmon tangent by $\underline{\alpha\beta}$ with an understroke; then we have, attending only to the magnitudes of the rectangles, by Art. 1, since S is touched by β, γ, δ', on one side, and by α on the other,

$$\underline{\alpha\beta}\cdot\gamma\delta' + \underline{\alpha\delta'}\cdot\beta\gamma - \underline{\alpha\gamma}\cdot\beta\delta' = 0; \qquad (a)$$

in like manner,

$$\underline{\alpha\gamma}\cdot\beta\delta' + \underline{\gamma\delta'}\cdot\alpha\beta - \underline{\beta\gamma}\cdot\alpha\delta' = 0; \qquad (b)$$

and

$$\underline{\alpha\beta}\cdot\gamma\delta' + \underline{\beta\delta'}\cdot\alpha\gamma - \underline{\beta\gamma}\cdot\alpha\delta' = 0. \qquad (c)$$

Hence, by adding equations (a) and (b), and subtracting equation (c), we get

$$\underline{\alpha\delta'}\cdot\beta\gamma + \underline{\gamma\delta'}\cdot\alpha\beta - \underline{\beta\delta'}\cdot\alpha\gamma = 0.$$

Hence the circles α, β, γ, δ', are all touched by a fourth circle having α, β, γ, on one side, and δ', on the other; hence we have the following system of eight circles :—

$\alpha, \beta, \gamma, \delta'$,	are all touched by a fourth circle.	(A)
$\alpha, \beta, \gamma', \delta$,	,,	(B)
$\alpha, \beta', \gamma, \delta$,	,,	(C)
$\alpha', \beta, \gamma, \delta$,	,,	(D)
$\alpha', \beta', \gamma', \delta$,	,,	(A′)
$\alpha', \beta', \gamma, \delta'$,	,,	(B′)
$\alpha', \beta, \gamma', \delta'$,	,,	(C′)
$\alpha, \beta', \gamma', \delta'$,	,,	(D′)

This proves the theorem for the system of eight circles; and from the foregoing scheme we see that the relation between the two systems of eight circles, α, β, γ, δ, α', β', γ', δ', and A, B, C, D, A', B', C', D', is reciprocal, viz., each circle of each system being touched by four circles of the other system—a property which was also noticed by Dr. Hart.—Q. E. D.

II.

EQUATIONS OF THE SIXTEEN SPHERES IN PAIRS WHICH TOUCH FOUR OTHERS.

15. If A, B, C, D, be four points in a plane, then denoting their three pairs of connectors by the following notation,

$$\begin{array}{llll} BC, & AD, & \text{by} & l^{\frac{1}{2}},\ p^{\frac{1}{2}}; \\ CA, & BD, & ,, & m^{\frac{1}{2}},\ q^{\frac{1}{2}}; \\ AB, & CD, & ,, & n^{\frac{1}{2}},\ r^{\frac{1}{2}}; \end{array}$$

we have (Salmon's "Geometry of Three Dimensions," Art. 50,)

$$\begin{aligned} & l(p-q)(p-r) + m(q-r)\,q-l) + n(r-p)(r-q) \\ & + lp(l-m-n) + mq(m-n-l) - nr(n-l-m) + lmn = 0. \qquad (24) \end{aligned}$$

This formula is the expansion of the following determinant:—

$$\begin{vmatrix} 0, & n, & m, & p, & 1, \\ n, & 0, & l, & q, & 1, \\ m, & l, & 0, & r, & 1, \\ p, & q, & r, & 0, & 1, \\ 1, & 1, & 1, & 1, & 0, \end{vmatrix} = 0. \qquad (25)$$

Now, if E be a fifth point in the plane or in space, and if A', B', C', D', be the points inverse to A, B, C, D, with respect to a

circle or sphere whose radius is K and centre at E, and denoting the connectors of the inverse points A', B', C', D', by the same notation as those of the points A, B, C, D, only with accents, also denoting EA', EB', EC', ED', by $\alpha^{\frac{1}{2}}$, $\beta^{\frac{1}{2}}$, $\gamma^{\frac{1}{2}}$, $\delta^{\frac{1}{2}}$,

Now it is evident that

$$l = \frac{K^4 l'}{\beta\gamma}, \quad m = \frac{K^4 m'}{\gamma\alpha}, \quad n = \frac{K^4 n'}{\alpha\beta},$$

$$p = \frac{K^4 p'}{\alpha\delta}, \quad q = \frac{K^4 r'}{\beta\delta}, \quad r = \frac{K^4 r'}{\gamma\delta}.$$

Hence, making these substitutions in equation (24), clearing of fractions, and omitting the accents as being no longer necessary, we have the following theorem:—

If A, B, C, D, E, be any five points in a plane, or on the surface of a sphere, and if the connectors in pairs of the points A, B, C, D, be denoted by $l^{\frac{1}{2}}, p^{\frac{1}{2}}$; $m^{\frac{1}{2}}, q^{\frac{1}{2}}$; $n^{\frac{1}{2}}, r^{\frac{1}{2}}$; and the connectors EA, EB, EC, ED, by $\alpha^{\frac{1}{2}}$, $\beta^{\frac{1}{2}}$, $\gamma^{\frac{1}{2}}$, $\delta^{\frac{1}{2}}$, then the relation

$$\begin{aligned} &lrq\alpha^2 + mpr\beta^2 + npq\gamma^2 + lmn\delta^2 + (lp - mq - nr)(l\alpha\delta + p\beta\gamma) \\ &+ (mq - nr - lp)(m\beta\delta + q\alpha\gamma) + (nr - lp - mq)(n\gamma\delta + r\alpha\beta) = 0, \quad (26) \end{aligned}$$

or its equivalent, the determinant,

$$\begin{vmatrix} 0, & n, & m, & p, & \alpha \\ n, & 0, & l, & q, & \beta \\ m, & l, & 0, & r, & \gamma \\ p, & q, & r, & 0, & \delta \\ \alpha, & \beta, & \gamma, & \delta, & 0 \end{vmatrix} = 0,$$

holds between these connectors.

16. The equation (26) between the connectors of five points on the surface of a sphere is the analogue of Ptolemy's theorem for four points on a circle, and can be enunciated in a very concise manner by the help of the following considerations:—

1°. The entire number of lines of connexion of five points $= \frac{5 \times 4}{1 \cdot 2} = 10$.

2°. The entire number of triangles which can be formed by combining the five points, three by three, is $\frac{5 \times 4 \times 3}{1 \cdot 2 \cdot 3} = 10$.

3°. The entire number of pentagons which can be formed having the five points for vertices $= \frac{4 \times 3 \times 2 \times 1}{2} = 12$.

Then, the sum of the ten products formed by multiplying the fourth power of the line joining any two points by the continued product of the squares of the sides of the triangle of which the three remaining points are vertices is equal to the sum of the twelve products formed, each, by multiplying together the squares of the sides of each of the twelve pentagons of which the five points are the vertices.

17. Supposing the points A, B, C, D, E, of Art. 14 to be on a plane, and that spheres whose diameters are $\delta, \delta', \delta'', \delta''', \delta''''$, touch the plane in those points; then inverting the whole from any arbitrary point in space, and denoting the common tangents to the inverse spheres by the same notation as that of Art. 14, viz., the common to the

inverse of the spheres at B, C, by $l^{\frac{1}{2}}$;

" " A, D, " $p^{\frac{1}{2}}$;

and so on; then we have from equation (21) the following theorem:—

If five spheres, A, B, C, D, E, touch a sixth sphere, Σ, the relation

$$\begin{vmatrix} 0, & n, & m, & p, & \alpha, \\ n, & 0, & l, & q, & \beta, \\ m, & l, & 0, & r, & \gamma, \\ p, & q, & r, & 0, & \delta, \\ \alpha, & \beta, & \gamma, & \delta, & 0, \end{vmatrix} = 0 \qquad (27)$$

holds between the common tangents of the five spheres, the common tangent to any pair of spheres being the direct or transverse, according as the pair of spheres to which it is drawn have contacts of the same or of opposite kinds with the sixth sphere, Σ.

18. The theorem of Art. 17 is an extension of the theorem of Art. 15, analogous to the extension which the theorem in Art. 1 is of Ptolemy's theorem, and an analogous use can be made of it.

For, supposing the sphere at the point E to reduce to a point, and denoting the other four spheres by S, S', S'', S''', then we get the equation of the pair of spheres touching $S, S,' S,'' S'''$,

$$\begin{vmatrix} 0, & n, & m, & p, & S \\ n, & 0, & l, & q, & S' \\ m, & l, & 0, & r, & S'' \\ p, & q, & r, & 0, & S''' \\ S, & S' & S'' & S''' & 0 \end{vmatrix} = 0, \qquad (28)$$

precisely in the same way as the equation of the pair of circles touching three circles was derived in Art. 2.—Q. E. D.

19. Denoting the equation (28), for shortness, by the notation $\phi(l, m, n, p, q, r,)=0$, and the transverse common tangents by the same notation as the direct common tangents, only with accents, and we have the following seven equations for the other seven pairs of spheres which touch S, S', S'', S''', viz.,

$$\phi(l, \; m', \; n', \; p', \; q, \; r,) = 0; \tag{29}$$
$$\phi(l', \; m, \; n', \; p, \; q', \; r,) = 0; \tag{30}$$
$$\phi(l', \; m', \; n, \; p, \; q, \; r',) = 0; \tag{31}$$
$$\phi(l, \; m, \; n, \; p', \; q', \; r',) = 0; \tag{32}$$
$$\phi(l', \; m', \; n, \; p', \; q', \; r,) = 0; \tag{33}$$
$$\phi(l, \; m', \; n', \; p, \; q', \; r',) = 0; \tag{34}$$
$$\phi(l', \; m, \; n', \; p', \; q, \; r',) = 0; \tag{35}$$

20. In precisely the same manner as in Art. 3 we derived the equations of the inscribed and exscribed circles of a plane triangle from the equations of the pairs of circles touching three circles, we can derive the equations of the eight spheres which touch the four faces of a tetrahedron from the equations of Arts. 18 and 19. Thus the equation of the inscribed sphere is, the faces being x, y, z, w, derived from equation (28)

$$\begin{vmatrix} 0, & \cos^2\frac{1}{2}(xy), & \cos^2\frac{1}{2}(xz), & \cos^2\frac{1}{2}(xw), & x \\ \cos^2\frac{1}{2}(yx), & 0, & \cos^2\frac{1}{2}(yz), & \cos^2\frac{1}{2}(yw), & y \\ \cos^2\frac{1}{2}(zx), & \cos^2\frac{1}{2}(zy), & 0, & \cos^2\frac{1}{2}(zw), & z \\ \cos^2\frac{1}{2}(wx), & \cos^2\frac{1}{2}(wy), & \cos^2\frac{1}{2}(wz) & 0, & w \\ x, & y, & z, & w, & 0 \end{vmatrix} = 0, \tag{36}$$

and the equations of the seven others are derived from equations (29)–(35).

21. Again, in the same way exactly as we derived the equations of the circles in pairs which touch three circles from the equations of the inscribed and exscribed circles of a plane triangle, we might derive the equations of the spheres in pairs which touch four spheres from the equations of the spheres touching the faces of a tetrahedron; and, in fact, it was in that way I first derived the theorem.

22. If we form the tangential equation corresponding to equation (23), we find—

$$\mu\nu l + \nu\lambda m + \lambda\mu n + \lambda\rho p + \mu\rho q + \nu\rho r = 0. \tag{37}$$

This is the condition that the pair of spheres given by equation (28) may be touched by the sphere $\lambda S + \mu S' + \nu S'' + \rho S''' = 0$. We get si-

milar equations from equations (29)–(35), inclusive; and since from any three of these equations we get eight systems of common values for λ, μ, ν, ρ, we infer that the three pairs of spheres denoted by any three of the equations (28)–(35) are touched by eight spheres, four of which are the spheres S, S', S'', S'''.

23. The eight tangential equations can all be included in one general formula, as follows:—

Let the radii of the four spheres, S, S', S'' S''', be denoted by r, r', r'', r''', and the angle at which S intersects S' by the notation (SS'), then we have

$$l = 4r'r'' \cos^2 \tfrac{1}{2}(S'S'');$$
$$l' = -4r'r'' \sin^2 \tfrac{1}{2}(S'S'');$$

and similar values for m, $m,'$ &c. Hence the equation (37) becomes transformed into

$$\begin{aligned}&\lambda\mu rr' \cos^2 \tfrac{1}{2}(SS') + \mu\nu r'r'' \cos^2 \tfrac{1}{2}(S'S'') + \nu\rho r''r''' \cos^2 \tfrac{1}{2}(S''S''')\\ &+ \lambda\nu rr'' \cos^2 \tfrac{1}{2}(SS'') + \lambda\rho rr''' \cos^2 \tfrac{1}{2}(SS''') + \mu\rho r'r''' \cos^2 \tfrac{1}{2}(S'S''').\end{aligned}$$

This is equivalent to the equation

$$U = (\lambda r + \mu r' + \nu r'' + \rho r''')^2, \tag{38}$$

where

$$\begin{aligned}U \equiv\ &\lambda^2 r^2 + \mu^2 r'^2 + \nu^2 r''^2 + \rho^2 r'''^2\\ &-2\lambda\mu rr' \cos(SS') - 2\mu\nu r'r'' \cos(S'S'') - 2\nu\rho r''r''' \cos(S''S''')\\ &-2\lambda\nu rr'' \cos(SS'') - 2\lambda\rho rr''' \cos(SS''') - 2\mu\rho r'r''' \cos(S'S''').\end{aligned} \tag{39}$$

And the eight tangential equations are included in the formula

$$U = (\lambda r \pm \mu r' \pm \nu r'' \pm \rho r''')^2 \tag{40}$$

(Compare Salmon's "Geometry of Three Dimensions," Art. 219.)

III.

EQUATIONS OF THE CIRCLES IN PAIRS WHICH TOUCH THREE CIRCLES ON A SPHERE.

24. The theorem, Art. 1, which was proved by inversion, can be proved without inversion, as follows:—

Let O, O', O'', O''', be the points of contact; A, B, C, D, the centres; r, r', r'', r''', the radii of four circles, S, S', S'', S''', which touch

a fifth circle, Σ; and let G be the centre, and R the radius of Σ; then we have (fig. 5),

$$4\sin^2\tfrac{1}{2}AGB = \frac{AB^2 - (BG - AG)^2}{AG \cdot GB} = \frac{AB^2 - (r - r')^2}{(R - r)(R - r')}$$

$$= \frac{\text{square of common tangent of } S, S'}{(R - r)(R - r')}.$$

Again,

$$4\sin^2\tfrac{1}{2}AGB = \frac{OO'^2}{R^2}.$$

Hence,

$$OO' = \text{common tangent of } S, S', \times \frac{R}{\sqrt{(R - r)(R - r')}}. \qquad (41)$$

Now, by Ptolemy's theorem,

$$OO' \cdot O''O''' + O'O'' \cdot OO''' + O''O \cdot O'O''' = 0.$$

Hence, substituting for OO', from (41), and making like substitutions for $O''O'''$, &c., we have the common tangent of S, S' by the common tangent of S'', $S''' +$ &c., $= 0$.

25. The proof given in the last Article is that alluded to in Art. 1; and it is evident that it may be proved in a manner precisely similar, if S, S', S'', S''', be four circles on the surface of a sphere touching a fifth circle, Σ, that the sin $\frac{1}{2}$common tangent of S, $S' +$ sin $\frac{1}{2}$ common tangent of S'', $S''' +$ sin $\frac{1}{2}$common tangent of S', $S'' \times$ sin $\frac{1}{2}$common tangent of S, $S''' +$ sin $\frac{1}{2}$common tangent of S'', $S \times$ sin $\frac{1}{2}$common tangent of S', $S''' = 0$, the common tangents being the direct or the transverse, according as the contacts of the pairs of circles to which they are drawn with Σ are similar or dissimilar.

26. The direct application of the theorem in the last two Articles gives at once a proof of Feuerbach's theorem for plane triangles, and of Dr. Hart's extension of it to spherical triangles.

For if S, S' S'', S''', be the inscribed and exscribed circles of a plane triangle, the common tangent of S, $S' = b - c$;

$$\text{of } S'', S''' = b + c.$$

Hence common tangent of S, $S' \times$ common tangent of S'', $S''' = b^2 - c^2$; and the other rectangles $= c^2 - a^2$, and $a^2 - b^2$, respectively. Hence the condition holds of S, S', S'', S''', being all touched by the same circle.—Q. E. D.

27. Again, if S, S', S'', S''', be the inscribed and exscribed circles of a spherical triangle, we have

$$\text{Sin}\,\tfrac{1}{2}\text{common tangent of } S, S' \times \sin\tfrac{1}{2}\text{common tangent of } S'', S''' = \sin^2\tfrac{1}{2}b - \sin^2\tfrac{1}{2}c;$$

and the other rectangles $= \sin^2\frac{1}{2}c - \sin^2\frac{1}{2}a$, and $\sin^2\frac{1}{2}a - \sin^2\frac{1}{2}b$, respectively. Hence the condition holds of the circles S, S', S'', S''', being all touched by the same circle.—Q. E. D.

28. It is evident that the three anharmonic ratios of the points of contact are

$$\frac{b^2-c^2}{c^2-a^2},\quad \frac{c^2-a^2}{a^2-b^2},\quad \frac{a^2-b^2}{b^2-c^2}, \tag{42}$$

for plane triangles; and for spherical triangles, they are

$$\frac{\sin^2\frac{1}{2}b-\sin^2\frac{1}{2}c}{\sin^2\frac{1}{2}c-\sin^2\frac{1}{2}a},\quad \frac{\sin^2\frac{1}{2}c-\sin^2\frac{1}{2}a}{\sin^2\frac{1}{2}a-\sin^2\frac{1}{2}b},\quad \frac{\sin^2\frac{1}{2}a-\sin^2\frac{1}{2}b}{\sin^2\frac{1}{2}b-\sin^2\frac{1}{2}c}. \tag{43}$$

29. Let P be the centre of a small circle S on the surface of a sphere (fig. 6); O a fixed point also on the surface, which we shall take as origin: OX a fixed great circle, corresponding to the initial line in plane geometry; and let $OP = n$, the angle $POX = m$, and the coordinates of any point Q of the circle S be ρ and θ, then we have from the spherical triangle OPQ, r being the radius of the circle S,

$$\cos r = \{\cos n \cos \rho + \sin n \sin \rho \cos (O - m)\} = 0. \tag{44}$$

This may be taken as the equation of the small circle S; and it is plain that with this system of co-ordinates the result of substituting the coordinates of any point Q' in the equation of a small circle S on the surface of a sphere is equal to

$$2\cos r \times \sin^2\tfrac{1}{2}\text{ the tangent from } Q' \text{ to } S.$$

This may be written

$$2\cos r \times \sin^2\tfrac{1}{2}t = S$$

$$\therefore \sin\tfrac{1}{2}t = \sqrt{\frac{S}{2\cos r}}. \tag{45}$$

30. If the small circle S''' of Art. 26 become a point, and if we denote the

$\sin\frac{1}{2}$ direct common tangent of S', S'', by $l^{\frac{1}{2}}$,

" $\quad S''$, S " $m^{\frac{1}{2}}$,

" $\quad S$, S' " $n^{\frac{1}{2}}$,

and the sines of half the transverse common tangents by $l'^{\frac{1}{2}}$, $m'^{\frac{1}{2}}$, $n'^{\frac{1}{2}}$, we get from the theorem of Art. 25, and from equation (45), the equations of the four pairs of circles which touch three small circles on the surface of a sphere, as follows :—

$$\sqrt{\frac{lS}{\cos r}} + \sqrt{\frac{mS'}{\cos r'}} + \sqrt{\frac{nS''}{\cos r''}} = 0\,; \tag{46}$$

$$\sqrt{\frac{lS}{\cos r}} + \sqrt{\frac{m'S'}{\cos r'}} + \sqrt{\frac{n'S''}{\cos r''}} = 0\,; \tag{47}$$

$$\sqrt{\frac{l'S}{\cos r}} + \sqrt{\frac{mS'}{\cos r'}} + \sqrt{\frac{n\,S''}{\cos r''}} = 0\,; \tag{48}$$

$$\sqrt{\frac{lS'}{\cos r}} + \sqrt{\frac{m'S'}{\cos r'}} + \sqrt{\frac{nS''}{\cos r''}} = 0. \tag{49}$$

31. The four equations, (46)–(49), when expanded, are equivalent to the following determinants:—

$$\begin{vmatrix} 0, & \dfrac{l}{\cos r'}, & \dfrac{m}{\cos r''}, & S \\ \dfrac{l}{\cos r'}, & 0, & \dfrac{n}{\cos r'''}, & S' \\ \dfrac{m}{\cos r'}, & \dfrac{n}{\cos r''}, & 0, & S'', \\ S, & S', & S'', & 0, \end{vmatrix} = 0. \ (50) \quad \begin{vmatrix} 0, & \dfrac{l}{\cos r}, & \dfrac{m'}{\cos r'}, & S, \\ \dfrac{l}{\cos r'}, & 0, & \dfrac{n'}{\cos r''}, & S', \\ \dfrac{m'}{\cos r''}, & \dfrac{n'}{\cos r''}, & 0, & S'', \\ S, & S', & S'', & 0, \end{vmatrix} = 0. \ (51)$$

$$\begin{vmatrix} 0, & \dfrac{l'}{\cos r'}, & \dfrac{m}{\cos r''}, & S, \\ \dfrac{l'}{\cos r'}, & 0, & \dfrac{n'}{\cos r'''}, & S'', \\ \dfrac{m}{\cos r''}, & \dfrac{n'}{\cos r''}, & 0, & S'', \\ S, & S', & S'', & 0, \end{vmatrix} = 0. \ (52) \quad \begin{vmatrix} 0, & \dfrac{l'}{\cos r}, & \dfrac{m'}{\cos r'}, & S, \\ \dfrac{l'}{\cos r}, & 0, & \dfrac{n}{\cos r''}, & S', \\ \dfrac{m'}{\cos r'}, & \dfrac{n'}{\cos r''}, & 0, & S'', \\ S, & S', & S', & 0, \end{vmatrix} = 0. \ (53)$$

And the four corresponding tangential equations are—

$$\frac{l}{\lambda \cos r} + \frac{m}{\mu \cos r'} + \frac{n}{\nu \cos r''} = 0; \tag{54}$$

$$\frac{l}{\lambda \cos r} + \frac{m'}{\mu \cos r'} + \frac{n'}{\nu \cos r''} = 0; \tag{55}$$

$$\frac{l'}{\lambda \cos r} + \frac{m}{\mu \cos r'} + \frac{n'}{\nu \cos r''} = 0; \tag{56}$$

$$\frac{l'}{\lambda \cos r} + \frac{m'}{\mu \cos r'} + \frac{n}{\nu \cos r''} = 0. \tag{57}$$

32. The proofs given in Articles 12 and 14 for Dr. Hart's extension of Feuerbach's theorem, it is evident, apply *verbatim* for the analogous theorem concerning circles on the sphere; and the part of it concerning the system of six circles, Art. 12, may also be inferred immediately from the equations (54)–(57); for any two of these equations are sufficient to determine the ratios $\lambda : \nu$ and $\mu : \nu$. Hence the four circles denoted by any pair of the equations (54)–(57) have a common tangential circle, besides the three circles S, S', S''.—Q. E. D.

33. The equations of the inscribed and exscribed circles of a spherical triangle may be inferred from equations (46)–(49).

For, denoting the angles at which they intersect

S'	and	S''	by	A;
S''	and	S	,,	B;
S	and	S'	,,	C;

it is easy to see that

$$2 \cos \tfrac{1}{2} A = \sqrt{\frac{l}{\tan r' \tan r''}}$$

$$2 \cos \tfrac{1}{2} B = \sqrt{\frac{m}{\tan r'' \tan r}}$$

$$2 \cos \tfrac{1}{2} C = \sqrt{\frac{n}{\tan r \tan r'}}$$

Hence equation (46) becomes transformed into

$$\cos \tfrac{1}{2} \sqrt{\frac{S}{\sin r}} + \cos \tfrac{1}{2} B \sqrt{\frac{S'}{\sin r'}} + \cos \tfrac{1}{2} C \sqrt{\frac{S''}{\sin r''}} = 0. \tag{58}$$

And if the circles S, S', S'', become great circles, denoting them by α, β, γ, we get for the equation of the inscribed circle of a spherical triangle—

$$\cos \tfrac{1}{2}A\sqrt{\alpha} + \cos^2 \tfrac{1}{2}B\sqrt{\beta} + \cos \tfrac{1}{2}C\sqrt{\gamma} = 0 ; \tag{59}$$

and the equations (47)-(49) give, when similarly transformed, the equations of the exscribed circles.

34. The tangential equations (54)-(57) become, by the substitutions of the last articles,

$$\frac{\cos^2 \frac{1}{2}A}{\lambda \sin r} + \frac{\cos^2 \frac{1}{2}B}{\mu \sin r'} + \frac{\cos^2 \frac{1}{2}C}{\nu \sin r''} = 0 ; \tag{60}$$

$$\frac{\cos^2 \frac{1}{2}A}{\lambda \sin r} - \frac{\sin^2 \frac{1}{2}B}{\mu \sin r'} - \frac{\sin^2 \frac{1}{2}C}{\nu \sin r''} = 0 ; \tag{61}$$

$$-\frac{\sin^2 \frac{1}{2}A}{\lambda \sin r} + \frac{\cos^2 \frac{1}{2}B}{\mu \sin r'} - \frac{\sin^2 \frac{1}{2}C}{\nu \sin r''} = 0 ; \tag{62}$$

$$-\frac{\sin^2 \frac{1}{2}A}{\lambda \sin r} - \frac{\sin^2 \frac{1}{2}B}{\mu \sin r'} + \frac{\cos^2 \frac{1}{2}C}{\nu \sin r''} = 0. \tag{63}$$

These formulæ are all included in the general formula

$$U = (\lambda \sin r \pm \mu \sin r' \pm \nu \sin r'')^2, \tag{64}$$

where

$$\begin{aligned} U \equiv\ & \lambda^2 \sin^2 r + \mu^2 \sin^2 r' + \nu^2 \sin^2 r'' \\ & - 2\mu\nu \sin r' \sin r'' \cos A - 2\nu\lambda \sin r'' \sin r \cos B \\ & - 2\lambda\mu \sin r \sin r' \cos C. \end{aligned} \tag{65}$$

35. The equations (60)-(63) denote the eight circles tangential to three circles on the sphere, and each pair are touched by the pair of circles

$$\begin{aligned} U = \{ & \lambda \sin r \cos (B - C) + \mu \sin r' \cos C - A) \\ & + \nu \sin r'' \cos (A - \mathrm{B}) \}^2 \end{aligned} \tag{66}$$

(See Salmon's "Geometry of Three Dimensions," Second Edition, Art. 253).

The pair of circles (66) correspond to the circles A, A' of Art. 14; and the circles corresponding to the pairs of circles B, B', C, C', D, D', of the same Article are

$$U = \{\lambda \sin r \cos (B - C) + \mu \sin r' \cos (C + A) \\ + \nu \sin r'' \cos (A + B\}^2, \quad (67)$$

$$U = \{\lambda \sin r \cos (B + C) + \mu \sin r' \cos C - A \\ + \nu \sin r'' \cos (A + B\}^2, \quad (68)$$

$$U = \{\lambda \sin r \cos (B + C) + \mu \sin r' \cos (C + A) \\ + \nu \sin r'' \cos (A - B)\}^2. \quad (69)$$

36. It can be seen precisely in the same way as in the analogous case on the plane (Art. 6), that the points of contact of three circles, S, S', S'', on the sphere with their four pairs of tangential circles are given by constructing the circles,

$$\frac{lS}{\cos r} = \frac{mS'}{\cos r} = \frac{nS''}{\cos r''}; \quad (70)$$

$$\frac{lS}{\cos r} = \frac{m'S'}{\cos r} = \frac{n'S''}{\cos r''}; \quad (71)$$

$$\frac{l'S}{\cos r} = \frac{mS'}{\cos r'} = \frac{n'S''}{\cos r''}; \quad (72)$$

$$\frac{l'S}{\cos r} = \frac{m'S'}{\cos r'} = \frac{nS''}{\cos r''}; \quad (73)$$

Also, that if A, A', A'', denote the radical circles of S, S', S'', taken in pairs, then the equations of the great circles passing through the points of contact on S, S', S'', are for the four pairs of tangential circles

$$\frac{A \cos r}{l} = \frac{A' \cos r'}{m} = \frac{A'' \cos r''}{n}; \quad (74)$$

$$\frac{A \cos r}{l} = \frac{A' \cos r'}{m'} = \frac{A'' \cos r''}{n'}; \quad (75)$$

$$\frac{A \cos r}{l'} = \frac{A' \cos r'}{m} = \frac{A'' \cos r''}{n'}; \quad (76)$$

$$\frac{A \cos r}{l'} = \frac{A' \cos r'}{m'} = \frac{A'' \cos r''}{n}. \quad (77)$$

37. Again, taking any pair of the circles S, S', S'', the four points of contact on it with any pair of the tangential circles (50)–(53) are concyclic; the equations of the circles passing through these concyclic points are, if we denote the determinants (50)–(53) by the notation ϕ_1, ϕ_2, ϕ_3, ϕ_4;

$$\frac{d\phi_1}{dS} = 0; \ \frac{d\phi_1}{dS'} = 0; \ \frac{d\phi_1}{dS''} = 0; \tag{78}$$

$$\frac{d\phi_2}{dS} = 0; \ \frac{d\phi_2}{dS'} = 0; \ \frac{d\phi_2}{dS''} = 0; \tag{79}$$

$$\frac{d\phi_3}{dS} = 0; \ \frac{d\phi_3}{dS'} = 0; \ \frac{d\phi_3}{dS''} = 0; \tag{80}$$

$$\frac{d\phi_4}{dS} = 0; \ \frac{d\phi_4}{dS'} = 0; \ \frac{d\phi_4}{dS''} = 0. \tag{81}$$

IV.

Equations of the Conics in Pairs having double Contact with a given Conic which touch three others also having double Contact with the same given Conic.

38. The equations of the circles on the surface of a sphere that we have employed hitherto denote but one of the intersections of a cone with a sphere whose centre is at the vertex of the cone; thus, if α, β, γ, be three such circles, then, taking account of the complete intersections of the sphere with the cones, it is evident we get three other circles, which we may denote by α', β', γ'; thus we have

8	circles touching	α, β, γ;
8	,,	α', β, γ;
8	,,	α, β', γ;
8	,,	α, β, γ';

hence we have 32 circles in all.

The equation $S - L^2 = 0$, of a small circle on the surface of the sphere, given in Dr. Salmon's "Geometry of Three Dimensions," is the complete intersection of the sphere with the cone; and it is easy to see that its factors $S^{\frac{1}{2}} - L = 0$ and $S^{\frac{1}{2}} + L = 0$ are the separate circles which make up the complete intersection; in fact, taking the equation of any small circle on the sphere,

$$\text{Cos}\, r - \{\cos n \cos \rho + \sin n \sin \rho \cos (\theta - m)\} = 0,$$

it is by transformation to three rectangular planes changed into

$$(x^2+y^2+z^2)^{\frac{1}{2}} \cos r = L,$$

where x, y, z, are the co-ordinates of any point in the circle, and L is the perpendicular from the same point on the plane of the great circle whose pole is the centre of the small circle; now, this equation is of the form

$$S^{\frac{1}{2}} - L = 0;$$

and it may be shown that the equation of its twin circle (see Salmon, page 200, foot note) is of the form

$$S^{\frac{1}{2}} + L = 0.$$

Hence the equation of the pair of circles touching three circles, α, β, γ, on the surface of a sphere may be written in either of the forms

$$\sqrt{\frac{l\,(S^{\frac{1}{2}}-L)}{\cos r}} + \sqrt{\frac{m\,(S^{\frac{1}{2}}-M)}{\cos r'}} + \sqrt{\frac{n\,(S^{\frac{1}{2}}-N)}{\cos r''}} = 0; \qquad (82)$$

$$\sqrt{\frac{l\,(S^{\frac{1}{2}}+L)}{\cos r}} + \sqrt{\frac{m\,(S^{\frac{1}{2}}+M)}{\cos r'}} + \sqrt{\frac{n\,(S^{\frac{1}{2}}-N)}{\cos r''}} = (4). \qquad (38)$$

And it will be seen that these, when cleared of radicals, give the same result; and the equations of the other pairs of circles are got from these by properly accenting l, m, n.

39. If S' S'', be two small circles of the sphere (fig. 7), $abcd$ a great circle passing through their centres, and J any circle cutting them orthogonally in a', b', c', d'; now, the anharmonic ratios of the four points, a', b', c', d', are equal to the anharmonic ratios of the points a, b, c, d; and two of the anharmonic ratios of the points a, b, c, d, are

$$\sin \tfrac{1}{2} ac \cdot \sin \tfrac{1}{2} bd : \sin \tfrac{1}{2} ab \cdot \sin \tfrac{1}{2} cd;$$

$$\sin \tfrac{1}{2} ad \cdot \sin \tfrac{1}{2} bc : \sin \tfrac{1}{2} ab \cdot \sin \tfrac{1}{2} cd;$$

and these are respectively equal to

$$l : \tan r' \cdot \tan r'';$$

$$l : \tan r' \cdot \tan r''.$$

Hence we have the following theorem :—

If any circle J cuts two small circles, S', S'', on the sphere orthogonally, two of the anharmonic ratios of the four points of section are

$$l : \tan r' \cdot \tan r'';$$

$$l' : \tan r' \cdot \tan r'';$$

where l, l', are the squares of the sines of half the direct and half the transverse common tangents of S', S''.

40. If the circles a, β, γ, be cut orthogonally by J, and denoting two of the anharmonic ratios (Art. 39) in which

J intersects	β, γ	by	λ, λ',
J	,,	γ, a	,,	μ, μ',
J	,,	a, β	,,	ν, ν',

we have

$$l = \lambda \tan r' \tan r'',$$

$$l = \lambda' \tan r' \tan r'';$$

and substituting this value of l and corresponding values for m and n in equation (70), it becomes transformed into

$$\sqrt{\frac{\lambda(S^{\frac{1}{2}} - L)}{\sin r}} + \sqrt{\frac{\mu(S^{\frac{1}{2}} - M)}{\sin r'}} + \sqrt{\frac{\nu(S^{\frac{1}{2}} - N)}{\sin r''}} = 0; \qquad (84)$$

and the equations of the other pairs of circles are got from this by properly accenting λ, μ, ν.

41. In equation (84) it will be observed that λ, μ, ν, are anharmonic ratios, and that $\sin r$, $\sin r'$, $\sin r''$, are the results of substituting the co-ordinates of the poles of the great circles L, M, N, in the equations

$$S - L^2 = 0; \ S - M^2 = 0; \ S - N^2 = 0.$$

These considerations will enable us to write down the equations of the conics in pairs having double contact with the conic S, and touching the three conics

$$S - L^2 = 0; \ S - M^2 = 0; \ S - N^2 = 0.$$

42. Since the equations $S - L^2 = 0$, $S - M^2 = 0$, $S - N^2 = 0$, are the same analytically as the equations of conics having double contact with a given conic, we can, by means of Arts. 40, 41, write down the equations of the conics in pairs having double contact with a given conic, and touching three others also having double contact with the same given conic. Thus, corresponding to the system of circles a, β, γ, we have the conics

$$S^{\frac{1}{2}} - L = 0, \ S^{\frac{1}{2}} - M = 0, \ S^{\frac{1}{2}} - N = 0,$$

whose common chords are (see Salmon, page 228)

$$L - M = 0, \ M - N = 0, \ N - L = 0.$$

Let these chords (fig. 8) intersect in O; then from O draw pairs of tangents to the conics; then it may be proved—but I shall not occupy space in doing so—that the six points of contact, a, a'; b, b'; c, c', are in the circumference of a conic, and denoting two of the anharmonic ratios, as in Art. 40, of the points

$$\begin{array}{lcl} a,\ a',\ b,\ b', & \text{by} & \lambda,\ \lambda'; \\ b,\ b',\ c,\ c', & ,, & \mu,\ \mu'; \\ c,\ c',\ a,\ a', & ,, & \nu,\ \nu'; \end{array}$$

and denoting the results of substituting the co-ordinates of the poles of the chords of contact L, M, N, in the equations of the conics $S - L^2 = 0$, $S - M^2 = 0$, $S - N^2 = 0$, by P, Q, R, respectively, we have from equation 84 the following system of equations of pairs of conics, each conic having double contact with S, and touching $S - L^2 = 0$, $S - M^2 = 0$, $S - N^2 = 0$:—

$$\sqrt{\frac{\lambda\,(S^{\frac{1}{2}} - L)}{P}} + \sqrt{\frac{\mu\,(S^{\frac{1}{2}} - M)}{Q}} + \sqrt{\frac{\nu\,(S^{\frac{1}{2}} - N)}{R}} = 0; \qquad (85)$$

$$\sqrt{\frac{\lambda\,(S^{\frac{1}{2}} - L)}{P}} + \sqrt{\frac{\mu'\,(S^{\frac{1}{2}} - M)}{Q}} + \sqrt{\frac{\nu'\,(S^{\frac{1}{2}} - N)}{R}} = 0; \qquad (86)$$

$$\sqrt{\frac{\lambda'\,(S^{\frac{1}{2}} - L)}{P}} + \sqrt{\frac{\mu\,(S^{\frac{1}{2}} - M)}{Q}} + \sqrt{\frac{\nu\,(S^{\frac{1}{2}} - N)}{R}} = 0; \qquad (87)$$

$$\sqrt{\frac{\lambda'\,(S^{\frac{1}{2}} - L)}{P}} + \sqrt{\frac{\mu'\,(S^{\frac{1}{2}} - M)}{Q}} + \sqrt{\frac{\nu\,(S^{\frac{1}{2}} - N)}{R}} = 0. \qquad (88)$$

43. The system of circles α', β, γ, have corresponding to them the system of conics $S^{\frac{1}{2}} + L = 0$, $S^{\frac{1}{2}} - M = 0$, $S^{\frac{1}{2}} - N = 0$; and denoting by $\lambda_1, \lambda'_1, \mu_1, \mu'_1, \nu_1, \nu'_1$, quantities analogous to $\lambda, \lambda', \mu, \mu', \nu, \nu'$, of the last article.

The common chords are $L + M = 0$, $M - N = 0$, $N + L = 0$; and the system of equations is

$$\sqrt{\frac{\lambda_1\,(S^{\frac{1}{2}} + L)}{P}} + \sqrt{\frac{\mu_1(S^{\frac{1}{2}} - M)}{Q}} + \sqrt{\frac{\nu_1\,(S^{\frac{1}{2}} - N)}{R}} = 0; \qquad (89)$$

$$\sqrt{\frac{\lambda_1\,(S^{\frac{1}{2}} + L)}{P}} + \sqrt{\frac{\mu'_1(S^{\frac{1}{2}} - M)}{Q}} + \sqrt{\frac{\nu'_1(S^{\frac{1}{2}} - N)}{R}} = 0; \qquad (90)$$

$$\sqrt{\frac{\lambda'_1(S^{\frac{1}{2}} + L)}{P}} + \sqrt{\frac{\mu_1\,(S^{\frac{1}{2}} - M)}{Q}} + \sqrt{\frac{\nu'_1(S^{\frac{1}{2}} - N)}{R}} = 0; \qquad (91)$$

$$\sqrt{\frac{\lambda'_1(S^{\frac{1}{2}} + L)}{P}} + \sqrt{\frac{\mu'_1\,(S^{\frac{1}{2}} - M)}{Q}} + \sqrt{\frac{\nu\,(S^{\frac{1}{2}} - N)}{R}} = 0; \qquad (92)$$

44. The system of circles α, β' γ, have corresponding to them the system of conics $S^{\frac{1}{2}} - L = 0$, $S^{\frac{1}{2}} + M = 0$, $S^{\frac{1}{2}} - N = 0$; common chords, $L + M = 0$, $M + N = 0$, $N - L = 0$; and let the anharmonic ratios be

$$\lambda_2, \lambda'_2, \mu_2, \mu'_2, \nu_2, \nu'_2,$$

then the corresponding system of equations is

$$\sqrt{\frac{\lambda_2 (S^{\frac{1}{2}} - L)}{P}} + \sqrt{\frac{\mu_2 (S^{\frac{1}{2}} + M)}{Q}} + \sqrt{\frac{\nu_2 (S^{\frac{1}{2}} - N)}{R}} = 0; \qquad (93)$$

$$\sqrt{\frac{\lambda_2 (S^{\frac{1}{2}} - L)}{P}} + \sqrt{\frac{\mu'_2 (S^{\frac{1}{2}} + M)}{Q}} + \sqrt{\frac{\nu'_2 (S^{\frac{1}{2}} - N)}{R}} = 0; \qquad (94)$$

$$\sqrt{\frac{\lambda'_2 (S^{\frac{1}{2}} - L)}{P}} + \sqrt{\frac{\mu_2 (S^{\frac{1}{2}} + M)}{Q}} + \sqrt{\frac{\nu'_2 (S^{\frac{1}{2}} - N)}{R}} = 0; \qquad (95)$$

$$\sqrt{\frac{\lambda'_2 (S^{\frac{1}{2}} + L)}{P}} + \sqrt{\frac{\mu'_2 (S^{\frac{1}{2}} + M)}{Q}} + \sqrt{\frac{\nu_2 (S^{\frac{1}{2}} + N)}{R}} = 0; \qquad (96)$$

45. The system of circles α, β, γ', have corresponding to them the system of conics $S^{\frac{1}{2}} - L = 0$, $S^{\frac{1}{2}} - M = 0$, $S^{\frac{1}{2}} + N = 0$; common chords, $L - M = 0$, $M + N = 0$, $N + L = 0$.

Let the anharmonic ratios be

$$\lambda_3, \lambda'_3, \mu_3, \mu'_3, \nu_3, \nu'_3;$$

then the corresponding system of equations is

$$\sqrt{\frac{\lambda_3 (S^{\frac{1}{2}} - M)}{P}} + \sqrt{\frac{\mu_3 (S^{\frac{1}{2}} - M)}{Q}} + \sqrt{\frac{\nu_3 (S^{\frac{1}{2}} + N)}{R}} = 0; \qquad (97)$$

$$\sqrt{\frac{\lambda_3 (S^{\frac{1}{2}} - L)}{P}} + \sqrt{\frac{\mu'_3 (S^{\frac{1}{2}} - M)}{Q}} + \sqrt{\frac{\nu'_3 (S^{\frac{1}{2}} + N)}{R}} = 0; \qquad (98)$$

$$\sqrt{\frac{\lambda'_3 (S^{\frac{1}{2}} - L)}{P}} + \sqrt{\frac{\mu_3 (S^{\frac{1}{2}} - M)}{Q}} + \sqrt{\frac{\nu'_3 (S^{\frac{1}{2}} + N)}{R}} = 0; \qquad (99)$$

$$\sqrt{\frac{\lambda'_3 (S^{\frac{1}{2}} - L)}{P}} + \sqrt{\frac{\mu'_3 (S^{\frac{1}{2}} - M)}{Q}} + \sqrt{\frac{\nu_3 (S^{\frac{1}{2}} + N)}{R}} = 0. \qquad (100)$$

46. We have in the last four articles given the equations of the sixteen pairs of conics having double contact with a given tonic, and touching three others also having double contact with the same given conic; and it is evident, from Articles 12, 14, 32, that Dr. Hart's theorem holds for each of the four systems into which we have written the equations.

47. It may be proved precisely as in Articles 7, 36, that the points of contact of the pair of conics given by the equation (85) with the conics $S-L^2-0$, $S-M^2=0$, $S-N=0$, are constructed by drawing the three lines

$$\frac{P(L-M)}{\lambda}=\frac{Q(M-N)}{\mu}=\frac{R(N-L)}{\nu}.$$

We give in the annexed scheme the entire system of forty-eight lines for the sixteen pairs of equations:—

1°. For the system of tangential conics (85)–(88) corresponding to the system of concurrent common chords

$$L-M=0\quad M-N=0,\ N-L=0,$$

the equations of the lines for constructing the points of contact are

$$\frac{P(L-M)}{\lambda}=\frac{Q(M-N)}{\mu}=\frac{R(N-L)}{\nu};\tag{101}$$

$$\frac{P(L-M)}{\lambda}=\frac{Q(M-N)}{\mu'}=\frac{R(N-L)}{\nu'};\tag{102}$$

$$\frac{P(L-M)}{\lambda'}=\frac{Q(M-N)}{\mu}=\frac{R(N-L)}{\nu'};\tag{103}$$

$$\frac{P(L-M)}{\lambda}=\frac{Q(M-N)}{\mu'}=\frac{R(N-L)}{\nu}.\tag{104}$$

2°. For the system of conics (89)–(92) corresponding to the system of concurrent common chords $L+M=0$, $M-N=0$, $N+L=0$, the equations of the lines through the points of contact are

$$\frac{P(L+M)}{\lambda_1}=\frac{Q(M-N)}{\mu_1}=\frac{R(N+L)}{\nu_1};\tag{105}$$

$$\frac{P(L+M)}{\lambda_1}=\frac{Q(M-N)}{\mu'_1}=\frac{R(N+L)}{\nu'_1};\tag{106}$$

$$\frac{P(L+M)}{\lambda'_1}=\frac{Q(M-N)}{\mu_1}=\frac{R(N+L)}{\nu'_1};\tag{107}$$

$$\frac{P(L+M)}{\lambda'_1}=\frac{Q(M-N)}{\mu'_1}=\frac{R(N+L)}{\nu_1}.\tag{108}$$

3°. For the system of conics (93)–(96) corresponding to the system of concurrent common chords $L+M=0$, $M+N=0$, $N-L=0$, of the conics $S-L^2=0$, $S-M^2=0$, $S-N^2=0$, the equations of the lines through the points of contact are,

$$\frac{P(L+M)}{\lambda_2} = \frac{Q(M+N)}{\mu_2} = \frac{R(N-L)}{\nu_2}; \qquad (109)$$

$$\frac{P(L+M)}{\lambda_2} = \frac{Q(M+N)}{\mu'_2} = \frac{R(N-L')}{\nu'_2}; \qquad (110)$$

$$\frac{P(L+M)}{\lambda'_2} = \frac{Q(M+N)}{\mu_2} = \frac{R(N-L)}{\nu'_2}; \qquad (111)$$

$$\frac{P(L+M)}{\lambda'_2} = \frac{Q(M+N)}{\mu'_2} = \frac{R(N-L)}{\nu_2}. \qquad (112)$$

4°. For the system of conics (97)-(100) corresponding to the system of concurrent common chords $L-M=0$, $M+N=0$, $N+L=0$, the lines through the points of contact are

$$\frac{P(L-M)}{\lambda_3} = \frac{Q(M+N)}{\mu_3} = \frac{R(N+L)}{\nu_3}; \qquad (113)$$

$$\frac{P(L-M)}{\lambda_3} = \frac{Q(M+N)}{\mu'_3} = \frac{R(N+L)}{\nu_3}; \qquad (114)$$

$$\frac{P(L-M)}{\lambda'_3} = \frac{Q(M+N)}{\mu_3} = \frac{R(N+L)}{\nu'^3}; \qquad (115)$$

$$\frac{P(L-M)}{\lambda'_3} = \frac{Q(M+N)}{\mu'_3} = \frac{R(N+L)}{\nu_3}. \qquad (116)$$

Hence we have a method of describing the sixteen pairs of conics.

In a subsequent paper I shall show that the greater number of the equations employed in this paper are capable of double interpretations, and also that the methods of demonstration employed can be used with advantage in other parts of geometry.

G. V. Du Noyer, Senior Geologist, G. S. I., M. R. I. A., presented the following collection of Drawings from original sketches of various antiquities, to form Vol. VII. of a similar donation to the Library of the Academy.

Early Irish and Pre-Norman Antiquities.

No. 1. Cromleac in the townland of Ballynageeragh, county of Waterford.

No. 2. Unfinished cromleac near Ballyphillip Bridge, Dunhill Glen, county of Waterford.*

* For detailed description of these cromleacs and remarks on the classification of ancient Irish earthen and megalithic structures, see a paper, by the same writer, in the "Kilkenny Archæological Journal" for April, 1866.